

Exact solutions and geometric phase factor of time-dependent three-generator quantum systems

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Abstract. There exist a number of typical and interesting systems and/or models, which possess three-generator Lie-algebraic structure, in atomic physics, quantum optics, nuclear physics and laser physics. The well-known fact that all simple 3-generator algebras are either isomorphic to the algebra $sl(2, C)$ or to one of its real forms enables us to treat these time-dependent quantum systems in a unified way. By making use of both the Lewis-Riesenfeld invariant theory and the invariant-related unitary transformation formulation, the present paper obtains exact solutions of the time-dependent Schrödinger equations governing various three-generator Lie-algebraic quantum systems. For some quantum systems whose *time-dependent* Hamiltonians have no quasialgebraic structures, it is shown that the exact solutions can also be obtained by working in a sub-Hilbert-space corresponding to a particular eigenvalue of the conserved generator (*i.e.*, the *time-independent* invariant that commutes with the *time-dependent* Hamiltonian). The topological property of geometric phase factors and its adiabatic limit in time-dependent systems is briefly discussed.

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1 Introduction

Exact solutions and geometric phase factor [1–4] of time-dependent spin model have been extensively investigated by many authors [5–9]. Bouchiat and Gibbons discussed the geometric phase for the spin-1 system [6]. Datta *et al.* found the exact solution for the spin- $\frac{1}{2}$ system [7] by means of the classical Lewis-Riesenfeld theory, and Mizrahi calculated the Aharonov-Anandan phase for the spin- $\frac{1}{2}$ system [8] in a time-dependent magnetic field. The more systematic approach to obtaining the formally exact solutions for the spin- j system was proposed by Gao *et al.* [9] who made use of the Lewis-Riesenfeld quantum theory [10]. In this spin- j system, the three Lie-algebraic generators of the Hamiltonian satisfy the commutation relations of $SU(2)$ Lie algebra. In addition to the spin model, there exist many quantum systems whose Hamiltonian is also constructed in terms of three generators of various Lie algebras, which we will illustrate in the following.

The invariant theory that can be applied to solutions of the time-dependent Schrödinger equation was first pro-

posed by Lewis and Riesenfeld in 1969 [10]. This theory is appropriate for treating the geometric phase factor. In 1991, Gao *et al.* generalized this theory and put forward the invariant-related unitary transformation formulation [11]. Exact solutions for time-dependent systems obtained by using the generalized invariant theory contain both the geometric phase and the dynamical phase [12–14]. This formulation was developed from the Lewis-Riesenfeld's formal theory and proven useful to the treatment of the exact solutions of the time-dependent Schrödinger equation and geometric phase factor. In the present paper, based on these invariant theories we obtain exact solutions of various time-dependent quantum systems with the three-generator Lie-algebraic structures.

This paper is organized as follows: in Section 2, we set out several quantum systems and models to illustrate the fact that many quantum systems and models possess three-generator Lie-algebraic structures; in Section 3, use is made of the invariant theories and exact solutions of various time-dependent three-generator systems are therefore obtained; in Section 4, there are some discussions concerning the closure property of the Lie-algebraic generators in the sub-Hilbert-space. In Section 5, we conclude this paper with some remarks.

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2 The algebraic structures of various three-generator quantum systems

In our previous work [15] we have shown that the time-dependent Schrödinger equation is solvable if its Hamiltonian is constructed in terms of the generators of a certain Lie algebra. This, therefore, implies that analyzing the algebraic structures of Hamiltonians plays significant role in obtaining exact solutions of time-dependent systems. To the best of our knowledge, a large number of quantum systems, which have three-generator Hamiltonians, have been considered in literature. Most of them, however, are considered only in *time-independent* cases, where the coefficients of the Hamiltonians are independent of time. In the present paper, we try to obtain the complete set of exact solutions of all these quantum systems in the *time-dependent* cases. In what follows we set out these systems and discuss the algebraic structures of their Hamiltonians.

2.1 Spin model

The time evolution of the wavefunction of a spinning particle in a magnetic field was studied by regarding it as a spin model [8] whose Hamiltonian can be written

$$H(t) = c_0 \left\{ \frac{1}{2} \sin \theta \exp[-i\varphi] J_+ + \frac{1}{2} \sin \theta \exp[i\varphi] J_- + \cos \theta J_3 \right\} \quad (1)$$

with $J_{\pm} = J_1 \pm iJ_2$ satisfying the commutation relations $[J_3, J_{\pm}] = \pm J_{\pm}$, $[J_+, J_-] = 2J_3$. Analogous to this case, in the gravitational theory of general relativity the Hamiltonians of both the spin-gravitomagnetic interaction [16] and the spin-rotation coupling [15,17,18] can be constructed in terms of J_+ , J_- , and J_3 . This, therefore, means that these interactions can be described by the spin model. It can be verified that the investigation of the propagation of a photon inside the noncoplanarly curved optical fiber [19–21] is also equivalent to that of a spin model. The Hamiltonian of spin model is composed of three generators which constitute the $SU(2)$ Lie algebra.

2.2 Two-coupled harmonic oscillator

The Hamiltonian of the two-coupled harmonic oscillator, which can describe the interaction of laser field with heat reservoirs [22], is of the form (in the unit $\hbar = 1$)

$$H = \omega_1 a_1^\dagger a_1 + \omega_2 a_2^\dagger a_2 + g a_1^\dagger a_2 + g^* a_2^\dagger a_1, \quad (2)$$

where $a_1^\dagger, a_2^\dagger, a_1, a_2$ are the creation and annihilation operators for these two harmonic oscillators, respectively; g and g^* are the coupling coefficients and g^* denotes the complex conjugation of g . Set $J_+ = a_1^\dagger a_2, J_- = a_2^\dagger a_1, J_3 = \frac{1}{2}(a_1^\dagger a_1 - a_2^\dagger a_2), N = \frac{1}{2}(a_1^\dagger a_1 + a_2^\dagger a_2)$, and

then one may show that the generators of this Hamiltonian represent the generators of the $SU(2)$ subalgebra in the Weyl-Heisenberg algebra. Since N commutes with H , *i.e.*, $[N, H] = 0$, we consequently say N is an invariant (namely, it is a conserved generator whose eigenvalue is time-independent). In terms of J_{\pm}, J_3 and N , the Hamiltonian in the expression (2) can be rewritten as follows

$$H = \omega_1(N + J_3) + \omega_2(N - J_3) + gJ_+ + g^*J_-. \quad (3)$$

Another interesting Hamiltonian of the two-coupled harmonic oscillator is written in the form

$$H = \omega_1 a_1^\dagger a_1 + \omega_2 a_2^\dagger a_2 + g a_1 a_2 + g^* a_1^\dagger a_2^\dagger, \quad (4)$$

which may describe the atomic dipole-dipole interaction without the rotating wave approximation [23]. If we take $K_+ = a_1^\dagger a_2^\dagger, K_- = a_1 a_2, K_3 = \frac{1}{2}(a_1 a_1^\dagger + a_2^\dagger a_2), N = \frac{1}{2}(a_1^\dagger a_1 - a_2^\dagger a_2)$, then the generators of the $SU(1, 1)$ group are thus realized. The commutation relations are immediately inferred as

$$[K_3, K_{\pm}] = \pm K_{\pm}, \quad [K_+, K_-] = -2K_3. \quad (5)$$

2.3 $SU(1, 1) \uplus_s \mathfrak{h}(4)$ Lie-algebra system

A good number of quantum systems whose Hamiltonian is some combinations of the generators of a Lie algebra, *e.g.*, $SU(1, 1) \uplus_s \mathfrak{h}(4)$ (\uplus_s denotes a semidirect sum) [24,25], which is used to discuss both the non-Poissonian effects in a laser-plasma scattering and the pulse propagation in a free-electron laser [25]. The $SU(1, 1) \uplus_s \mathfrak{h}(4)$ Hamiltonian is

$$H = AK_3 + FK_+ + F^*K_- + Ba^\dagger + B^*a + G, \quad (6)$$

where a and a^\dagger are harmonic-oscillator annihilation and creation operators, respectively.

2.4 General harmonic oscillator

The Hamiltonian of the general harmonic oscillator is given by [11]

$$H = \frac{1}{2}[Xq^2 + Y(qp + pq) + Zp^2] + Fq, \quad (7)$$

where the canonical coordinate q and the canonical momentum p satisfy the commutation relation $[q, p] = i$. The following three-generator Lie algebra is easily derived

$$\begin{aligned} [q^2, p^2] &= 2i(qp + pq), & [i(qp + pq), q^2] &= 4q^2, \\ [i(qp + pq), p^2] &= -4p^2. \end{aligned} \quad (8)$$

2.5 Charged particle moving in a magnetic field

The motion of a particle with mass μ and charge e in a homogeneous magnetic field $\mathbf{B} = (0, 0, B)$ is described by the following Hamiltonian in the spherical coordinates

$$H = -\frac{1}{2\mu} \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} L^2 \right) + \frac{1}{8} \mu \omega^2 r^2 - \frac{\omega}{2} L_z \quad (9)$$

with $\omega = B/\mu$, where L^2 and L_z respectively denote the square and the third component of the angular momentum operator of the particle moving in the magnetic field. Since both L_z and L^2 commute with H and thus they are called invariants, only the operators associated with r should be taken into consideration. We can show that if the following operators are defined

$$\begin{aligned} K_1 &= \mu r^2, & K_2 &= -\frac{1}{\mu} \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} L^2 \right), \\ K_3 &= -2i \left(\frac{3}{2} + r \frac{\partial}{\partial r} \right), \end{aligned} \quad (10)$$

then K_1, K_2 and K_3 form an algebra

$$\begin{aligned} [K_1, K_2] &= 2iK_3, & [K_3, K_2] &= 4iK_2, \\ [K_3, K_1] &= -4iK_1. \end{aligned} \quad (11)$$

Apparently, H can be rewritten in terms of the generators of this Lie algebra.

2.6 Two-level atomic coupling

The model under consideration is consisted of two-level atom driven by the photons field [26]. The interaction part of the Hamiltonian contains the transition operator $|1\rangle\langle 2|$ and $|2\rangle\langle 1|$, where $|1\rangle$ and $|2\rangle$ are the atomic operators of the two-level atom. Simple calculation yields

$$\begin{aligned} [|1\rangle\langle 1| - |2\rangle\langle 2|, |1\rangle\langle 2|] &= 2|1\rangle\langle 2|, \\ [|1\rangle\langle 1| - |2\rangle\langle 2|, |2\rangle\langle 1|] &= -2|2\rangle\langle 1|, \\ [|1\rangle\langle 2|, |2\rangle\langle 1|] &= |1\rangle\langle 1| - |2\rangle\langle 2|, \end{aligned} \quad (12)$$

which unfolds that the Hamiltonian contains a $SU(2)$ algebraic structure.

2.7 Supersymmetric Jaynes-Cummings model

In addition to the ordinary Jaynes-Cummings models [27], there exists a two-level multiphoton Jaynes-Cummings (TLMJC) model which possesses supersymmetric structure. In this generalization of the Jaynes-Cummings model, the atomic transitions are mediated by k photons [28–30]. Singh has shown that this model can be used to study multiple atom scattering of radiation and multiphoton emission, absorption, and laser processes [31]. The Hamiltonian of this model under the rotating wave approximation is given by

$$H(t) = \omega(t)a^\dagger a + \frac{\omega_0(t)}{2}\sigma_z + g(t)(a^\dagger)^k \sigma_- + g^*(t)a^k \sigma_+, \quad (13)$$

where a^\dagger and a are the creation and annihilation operators for the electromagnetic field, and obey the commutation relation $[a, a^\dagger] = 1$; σ_\pm and σ_z denote the two-level atom operators which satisfy the commutation relation $[\sigma_z, \sigma_\pm] = \pm 2\sigma_\pm$. We can verify that this model is solvable and the complete set of exact solutions can be found

by working in a sub-Hilbert-space corresponding to a particular eigenvalue of the supersymmetric generator N'

$$N' = \begin{pmatrix} a^k (a^\dagger)^k & 0 \\ 0 & (a^\dagger)^k a^k \end{pmatrix}. \quad (14)$$

It can be verified that N' commutes with the Hamiltonian in (13), and N' is therefore called the time-independent invariant. the commutation relations of its supersymmetric Lie-algebraic structure are

$$\begin{aligned} [Q^\dagger, Q] &= \lambda_m \sigma_z, & [N, Q] &= Q, & [N, Q^\dagger] &= -Q^\dagger, \\ [Q, \sigma_z] &= 2Q, & [Q^\dagger, \sigma_z] &= -2Q^\dagger, \end{aligned} \quad (15)$$

where

$$N = a^\dagger a + \frac{k-1}{2}\sigma_z + \frac{1}{2}, \quad Q = (a^\dagger)^k \sigma_-, \quad Q^\dagger = a^k \sigma_+, \quad (16)$$

and $\lambda_m = (m+k)!/m!$ denotes the eigenvalue of the time-independent invariant N' with the eigenvalue equation being

$$N' \begin{pmatrix} |m\rangle \\ |m+k\rangle \end{pmatrix} = \lambda_m \begin{pmatrix} |m\rangle \\ |m+k\rangle \end{pmatrix}. \quad (17)$$

By the aid of (15) and (16), the Hamiltonian (13) of this supersymmetric Jaynes-Cummings model can be rewritten as

$$H(t) = \omega(t)N + \frac{\omega(t) - \delta(t)}{2}\sigma_z + g(t)Q + g^*(t)Q^\dagger - \frac{\omega(t)}{2}. \quad (18)$$

with $\delta(t) = k\omega(t) - \omega_0(t)$.

Vogel and Welsch have studied the k -photon Jaynes-Cummings model with coherent atomic preparation which is time-independent [32]. In the framework of the formulation presented in this paper, we can study the totally time-dependent cases of work done by Vogel and Welsch [32].

2.8 Two-level atom interacting with a generalized cavity

Consider the following Hamiltonian [33]

$$H = r(A_0) + s(A_0)\sigma_z + gA_- \sigma_+ + g^*A_+ \sigma_- \quad (19)$$

where $r(A_0)$ and $s(A_0)$ are well-defined real functions of A_0 , and A_0, A_\pm satisfy the commutation relations $[A_0, A_\pm] = \pm mA_\pm$ [34]. One can show that this Hamiltonian possesses a three-generator algebraic structure.

2.9 The interaction between a hydrogenlike atom and an external magnetic field

This model is described by

$$\begin{aligned} H &= \alpha \mathbf{L} \cdot \mathbf{S} + \beta(L_z + 2S_z) \\ &= \beta L_z + \left(\frac{1}{2} \alpha L_z + \beta \right) \sigma_z + \frac{1}{2} \alpha (L_- \sigma_+ + L_+ \sigma_-) \end{aligned} \quad (20)$$

with $L_{\pm} = L_x \pm iL_y$. It is evidently seen that this form of Hamiltonian is analogous to that in (19).

2.10 Coupled two-photon lasers

The Hamiltonian of coupled two-photon lasers is in fact the combination of the two Hamiltonians of the two-coupled harmonic oscillator and general harmonic oscillator. One can show that there exists a $SU(2)$ algebraic structure in this model [35].

From what has been discussed above we can draw a conclusion that a number of typical and useful systems and models in laser physics, atomic physics and quantum optics can be attributed to various three-generator types. Dattoli *et al.* have also studied the Lie-algebraic structures and time evolutions of most of the above illustrative examples [36]. It should be noted that most of above systems and models in literature were only considered in stationary cases where the coefficients of the Hamiltonians were totally time-independent (or partly time-dependent). In the present paper, we will further indicate that the analysis of the algebraic structures shows the solvability of these quantum systems. In the meanwhile we give exact solutions of the time-dependent Schrödinger equation of all these systems and models where all the coefficients of the Hamiltonians are time-dependent.

3 Exact solutions of time-dependent Schrödinger equation

Time evolution of most above systems and models is governed by the Schrödinger equation

$$i \frac{\partial |\Psi(t)\rangle_s}{\partial t} = H(t) |\Psi(t)\rangle_s, \quad (21)$$

where the Hamiltonian is constructed by three generators A, B and C and is often given as follows

$$\begin{aligned} H(t) &= \omega(t) \left\{ \frac{1}{2} \sin \theta(t) \exp[-i\phi(t)] A \right. \\ &\quad \left. + \frac{1}{2} \sin \theta(t) \exp[i\phi(t)] B + \cos \theta(t) C \right\} \end{aligned} \quad (22)$$

with A, B and C satisfying the general commutation relations of a Lie algebra

$$[A, B] = nC, \quad [C, A] = mA, \quad [C, B] = -mB, \quad (23)$$

where m and n are the structure constants of this Lie algebra. Here, for convenience, the Hamiltonians with the three-generator Lie-algebraic structures is parameterized to be expression (22) in terms of the parameters ω, θ and ϕ . For instance, in model Hamiltonian (3), the Lie-algebraic generators, A, B and C in expression (22), may respectively stand for J_+, J_- , and J_z . Thus the parameters ω, θ and ϕ may be determined by

$$\begin{aligned} g(t) &= \frac{1}{2} \omega(t) \sin \theta(t) \exp[-i\phi(t)], \\ g^*(t) &= \frac{1}{2} \omega(t) \sin \theta(t) \exp[i\phi(t)], \\ \omega_1(t) - \omega_2(t) &= \omega(t) \cos \theta(t), \end{aligned} \quad (24)$$

with $\omega(t) = \sqrt{[\omega_1(t) - \omega_2(t)]^2 + 4g(t)g^*(t)}$. The same parameterizing approach is also readily applied to the supersymmetric Jaynes-Cummings model Hamiltonian (18) and all other three-generator Lie-algebraic quantum systems and models presented in this paper. Since all simple 3-generator algebras are either isomorphic to the algebra $sl(2, C)$ or to one of its real forms, we treat these time-dependent quantum systems in a unified way. According to the Lewis-Riesenfeld invariant theory, an operator $I(t)$ that agrees with the following invariant equation [10]

$$\frac{\partial I(t)}{\partial t} + \frac{1}{i} [I(t), H(t)] = 0 \quad (25)$$

is called an invariant whose eigenvalue is time-independent, *i.e.*,

$$I(t) |\lambda, t\rangle_I = \lambda |\lambda, t\rangle_I, \quad \frac{\partial \lambda}{\partial t} = 0. \quad (26)$$

It is seen from equation (25) that $I(t)$ is the linear combination of A, B and C and may be generally written

$$\begin{aligned} I(t) &= y \left\{ \frac{1}{2} \sin a(t) \exp[-ib(t)] A + \frac{1}{2} \sin a(t) \exp[ib(t)] B \right\} \\ &\quad + \cos a(t) C, \end{aligned} \quad (27)$$

where the constant y will be determined below. It should be pointed out that it is not the only way to construct the invariants. Since the product of two invariants also satisfies equation (25) [11], there are infinite invariants of a time-dependent quantum system. But the form in equation (27) is the most convenient and useful one. Substitution of (27) into equation (25) yields

$$\begin{aligned} y \exp(-ib) (\dot{a} \cos a - \dot{b} \sin a) - im\omega [\exp(-i\phi) \cos a \sin \theta \\ - y \exp(-ib) \sin a \cos \theta] = 0, \\ \dot{a} + \frac{ny}{2} \omega \sin \theta \sin(b - \phi) = 0. \end{aligned} \quad (28)$$

where dot denotes the time derivative. The time-dependent parameters a and b are determined by these two auxiliary equations.

It is easy to verify that the particular solution $|\Psi(t)\rangle_s$ of the Schrödinger equation can be expressed in terms of the eigenstate $|\lambda, t\rangle_I$ of the invariant $I(t)$, namely,

$$|\Psi(t)\rangle_s = \exp\left[\frac{1}{i}\varphi(t)\right] |\lambda, t\rangle_I \quad (29)$$

with

$$\varphi(t) = \int_{0I}^t \langle \lambda, t' | [H(t') - i\frac{\partial}{\partial t'}] |\lambda, t'\rangle_I dt'. \quad (30)$$

The physical meanings of

$$\int_{0I}^t \langle \lambda, t' | H(t') |\lambda, t'\rangle_I dt'$$

and

$$\int_{0I}^t \langle \lambda, t' | -i\frac{\partial}{\partial t'} |\lambda, t'\rangle_I dt'$$

are dynamical and geometric phase, respectively.

Since the expression (29) is merely a formal solution of the Schrödinger equation, it is of no use to investigate the time evolution and geometric effects (topological and global properties) of time-dependent quantum systems. In order to get the explicit solutions we make use of the invariant-related unitary transformation formulation [11], which enables one to obtain the complete set of exact solutions of the time-dependent Schrödinger equation (21). In accordance with the invariant-related unitary transformation method, the time-dependent unitary transformation operator is often of the form

$$V(t) = \exp[\beta(t)A - \beta^*(t)B] \quad (31)$$

with

$$\beta(t) = -\frac{a(t)}{2}x \exp[-ib(t)],$$

$$\beta^*(t) = -\frac{a(t)}{2}x \exp[ib(t)],$$

where the constant, x , will be determined below. By making use of the Glauber formula, lengthy calculation yields

$$\begin{aligned} I_V &= V^\dagger(t)I(t)V(t) \\ &= \left\{ \frac{y}{2} \exp(-ib) \sin a \cos \left[\left(\frac{mn}{2} \right)^{\frac{1}{2}} ax \right] \right. \\ &\quad \left. - \frac{\left(\frac{mn}{2} \right)^{\frac{1}{2}}}{n} \exp(-ib) \cos a \sin \left[\left(\frac{mn}{2} \right)^{\frac{1}{2}} ax \right] \right\} A \\ &\quad + \left\{ \frac{y}{2} \exp(ib) \sin a \cos \left[\left(\frac{mn}{2} \right)^{\frac{1}{2}} ax \right] \right. \\ &\quad \left. - \frac{\left(\frac{mn}{2} \right)^{\frac{1}{2}}}{n} \exp(ib) \cos a \sin \left[\left(\frac{mn}{2} \right)^{\frac{1}{2}} ax \right] \right\} B \\ &\quad + \left\{ \cos a \cos \left[\left(\frac{mn}{2} \right)^{\frac{1}{2}} ax \right] + \frac{\left(\frac{mn}{2} \right)^{\frac{1}{2}}}{m} y \right. \\ &\quad \left. \times \sin a \sin \left[\left(\frac{mn}{2} \right)^{\frac{1}{2}} ax \right] \right\} C. \end{aligned} \quad (32)$$

It can be easily seen that when y and x are taken to be

$$y = \frac{m}{\left(\frac{mn}{2} \right)^{\frac{1}{2}}}, \quad x = \frac{1}{\left(\frac{mn}{2} \right)^{\frac{1}{2}}}, \quad (33)$$

one may derive that $I_V = C$, which is time-independent. Thus the eigenvalue equation of the time-independent invariant I_V may be written in the form

$$I_V |\lambda\rangle = \lambda |\lambda\rangle, \quad |\lambda\rangle = V^\dagger(t) |\lambda, t\rangle_I. \quad (34)$$

Under the transformation $V(t)$, the Hamiltonian $H(t)$ can be changed into

$$\begin{aligned} H_V(t) &= V^\dagger(t)H(t)V(t) - V^\dagger(t)i\frac{\partial V(t)}{\partial t} \\ &= \left\{ \omega \left[\cos a \cos \theta + \frac{\left(\frac{mn}{2} \right)^{\frac{1}{2}}}{m} \sin a \sin \theta \cos(b - \phi) \right] \right. \\ &\quad \left. + \frac{\dot{b}}{m}(1 - \cos a) \right\} C \end{aligned} \quad (35)$$

by the aid of Baker-Campbell-Hausdorff formula [37]

$$\begin{aligned} V^\dagger(t)\frac{\partial}{\partial t}V(t) &= \frac{\partial}{\partial t}L + \frac{1}{2!} \left[\frac{\partial}{\partial t}L, L \right] + \frac{1}{3!} \left[\left[\frac{\partial}{\partial t}L, L \right], L \right] \\ &\quad + \frac{1}{4!} \left[\left[\left[\frac{\partial}{\partial t}L, L \right], L \right], L \right] + \dots \end{aligned} \quad (36)$$

with $V(t) = \exp[L(t)]$. Hence, with the help of equations (29, 34), the particular solution of the Schrödinger equation is obtained

$$|\Psi(t)\rangle_s = \exp\left[\frac{1}{i}\varphi(t)\right] V(t) |\lambda\rangle \quad (37)$$

with the phase

$$\begin{aligned} \varphi(t) &= \int_0^t \langle \lambda | \left[V^\dagger(t')H(t')V(t') - V^\dagger(t')i\frac{\partial}{\partial t'}V(t') \right] |\lambda\rangle dt' \\ &= \varphi_d(t) + \varphi_g(t) \\ &= \lambda \int_0^t \left\{ \omega \left[\cos a \cos \theta + \frac{\left(\frac{mn}{2} \right)^{\frac{1}{2}}}{m} \sin a \sin \theta \cos(b - \phi) \right] \right. \\ &\quad \left. + \frac{\dot{b}}{m}(1 - \cos a) \right\} dt', \end{aligned} \quad (38)$$

where the dynamical phase is

$$\begin{aligned} \varphi_d(t) &= \lambda \int_0^t \omega \left[\cos a \cos \theta \right. \\ &\quad \left. + \frac{\left(\frac{mn}{2} \right)^{\frac{1}{2}}}{m} \sin a \sin \theta \cos(b - \phi) \right] dt' \end{aligned}$$

and the geometric phase is

$$\varphi_g(t) = \lambda \int_0^t \frac{\dot{b}}{m}(1 - \cos a) dt'.$$

It is seen that the former phase is related to the coefficients of the Hamiltonian such as ω , $\cos \theta$, $\sin \theta$, etc., whereas the latter is not immediately related to these coefficients. If the parameter a is taken to be time-independent,

$$\varphi_g(T) = \lambda \int_0^T \frac{\dot{b}}{m} (1 - \cos a) dt' = \frac{\lambda}{m} [2\pi(1 - \cos a)]$$

where $2\pi(1 - \cos a)$ is an expression for the solid angle over the parameter space of the invariant. It is of interest that $(\lambda/m)[2\pi(1 - \cos a)]$ is equal to the magnetic flux produced by a magnetic monopole (and the gravitomagnetic monopole) of strength $\lambda/4\pi m$ existing at the origin of the parameter space [38]. This, therefore, implies that geometric phase differs from dynamical phase and it involves the global and topological properties of the time evolution of a quantum system. This fact indicates the geometric and topological meaning of $\varphi_g(t)$.

Here we briefly concern ourselves with the model Hamiltonians such as (3) and (4) with the conserved generator (*i.e.*, the time-independent invariant of which the eigenvalue is time-independent). For the model Hamiltonian (3), it can be rewritten as follows

$$H(t) = g(t)J_+ + g^*(t)J_- + [\omega_1(t) - \omega_2(t)] J_3 + [\omega_1(t) + \omega_2(t)] N \quad (39)$$

with $N = (a_1^\dagger a_1 + a_2^\dagger a_2)/2$ being the time-independent invariant that satisfies the commutation relation $[N, H(t)] = 0$ (*i.e.*, N commutes with the time-dependent Hamiltonian $H(t)$). Since N is an invariant, the eigenvalue of N may be $(n_1 + n_2)/2$, where n_1 and n_2 denote the eigenvalue of $a_1^\dagger a_1$ and $a_2^\dagger a_2$, respectively. In Section 4, we will show that a generalized quasialgebra can be found by working in a sub-Hilbert-space corresponding to a particular eigenvalue, $(n_1 + n_2)/2$, of the time-independent invariant N , where N can be replaced with the particular eigenvalue, $(n_1 + n_2)/2$, in the Lie-algebraic commutation relations. We thus rewrite the model Hamiltonian (3) as follows

$$H(t) = g(t)J_+ + g^*(t)J_- + [\omega_1(t) - \omega_2(t)] J_3 + \frac{1}{2} (n_1 + n_2) [\omega_1(t) + \omega_2(t)]. \quad (40)$$

It is easily seen that this form of Hamiltonian in the sub-Hilbert-space is different from that in (22) only by a time-dependent c -numbers $(n_1 + n_2)[\omega_1(t) + \omega_2(t)]/2$, which contributes only a time-dependent dynamic phase factor expressed by $\exp\{\int_0^t \frac{1}{2}(n_1 + n_2)[\omega_1(t') + \omega_2(t')] dt'\}$ to the particular solution of the time-dependent Schrödinger equation. So, the method presented above can be readily applied to the model Hamiltonian (3). In the same fashion, the model Hamiltonian (4) can be rewritten

$$H(t) = g(t)K_- + g^*(t)K_+ + [\omega_1(t) + \omega_2(t)] K_3 + [\omega_1(t) - \omega_2(t)] N - \frac{1}{2} [\omega_1(t) + \omega_2(t)] \quad (41)$$

with $N = (a_1^\dagger a_1 - a_2^\dagger a_2)/2$ being the time-independent invariant that commutes with this time-dependent Hamiltonian $H(t)$ (41). Since N is an invariant, in the sub-Hilbert-space corresponding to the particular eigenvalue, $(n_1 - n_2)/2$, of N , the Hamiltonian (4) may be rewritten as follows

$$H(t) = g(t)K_- + g^*(t)K_+ + [\omega_1(t) + \omega_2(t)] K_3 + \frac{1}{2} (n_1 - n_2) [\omega_1(t) - \omega_2(t)] - \frac{1}{2} [\omega_1(t) + \omega_2(t)], \quad (42)$$

which differs from the Hamiltonian (22) only by a c -numbers $(n_1 - n_2)[\omega_1(t) - \omega_2(t)]/2 - [\omega_1(t) + \omega_2(t)]/2$ that also contributes only a time-dependent dynamic phase factor to the particular solution of the time-dependent Schrödinger equation. Hence we can obtain the solutions of this class of model with the conserved generator (*i.e.*, the time-independent invariant) by working in a sub-Hilbert-space corresponding to a particular eigenvalue of the time-independent invariant. Maamache informed us of that he had obtained the exact solutions of these two quantum models [39] also by using the invariant formulation. But there is no so-called conserved generator in Maamache's model Hamiltonians. It is verified that when the general results equations (37, 38) (deducting the dynamic phase factor associated with the conserved generators) are applied to these two model, the solutions obtained are in complete agreement with Maamache's results. The more complicated cases such as the supersymmetric Jaynes-Cummings model whose Hamiltonian possesses the conserved generator are discussed in more detail in the next section.

To conclude this section, we briefly discuss the concepts of the exact solution and the explicit solution. The expression (37) is a particular exact solution corresponding to the eigenvalue λ of the invariant, and the general solutions of the time-dependent Schrödinger equation are therefore easily obtained by using the linear combinations of all these particular solutions. Generally speaking, in Quantum Mechanics, solution with chronological-product operator (time-order operator) P is often called the formal solution. In the present paper, however, the solution of the Schrödinger equation governing a time-dependent system is sometimes called the explicit solution, for reasons that the solution does not involve time-order operator. But, on the other hand, by using the Lewis-Riesenfeld invariant theory, there always exist some time-dependent parameters, *e.g.*, $a(t)$ and $b(t)$ in this paper which are determined by the auxiliary equations (28). According to the traditional practice, when employed in experimental analysis and compared with experimental results, these nonlinear auxiliary equations should be solved often by means of numerical calculation. From above viewpoints, the concept of explicit solution is understood in a relative sense, namely, it can be considered the explicit solution when compared with the time-evolution operator $U(t) = P \exp[(1/i) \int_0^t H(t') dt']$ involving the time-order operator, P ; whereas, it cannot be considered completely the explicit solution for it is expressed in terms of some time-dependent parameters, which should be obtained

via the auxiliary equations. Hence, conservatively speaking, the solution of the time-dependent system presented in the paper is often regarded as the exact solution rather than the explicit solution.

4 Exact solutions obtained in the sub-Hilbert-space

In the previous section, we obtain exact solutions of some time-dependent three-generator systems and models possessing the three-generator Lie-algebraic structures by using these invariant theories. However, there exists a problem of the closure property of the Lie-algebraic generators in the sub-Hilbert-space, which will be further discussed in what follows.

The generalized invariant theory can only be applied to the treatment of the system for which there exists the quasialgebra defined in reference [8]. Unfortunately, it is seen from (19) and (20) that there is no such quasialgebra in the Hamiltonians of example (7) (*i.e.*, the supersymmetric Jaynes-Cummings model) and example (8) (*i.e.*, the two-level atom interacting with a generalized cavity). In order to solve these two models, we generalize the method that has been used for finding the dynamical algebra $O(4)$ of the hydrogen atom to treat the Hamiltonians of these time-dependent models. In the case of hydrogen, the dynamical algebra $O(4)$ was found by working in the sub-Hilbert-space corresponding to a particular eigenvalue of the Hamiltonian [40]. In this paper, we will show that a generalized quasialgebra can also be found by working in a sub-Hilbert-space corresponding to a particular eigenvalue of the operator $\Delta = A_0 + m(1 + \sigma_z)/2$ in the time-dependent model of two-level atom interacting with a generalized cavity. This generalized quasialgebra enables one to obtain the complete set of exact solutions for the Schrödinger equation. It is readily verified that the operator Δ commutes with $H(t)$ and is therefore a time-independent invariant according to equation (25). In order to unfold the algebraic structure of the Hamiltonian (19), the following three operators are defined [33]

$$\begin{aligned}\Sigma_1 &= \frac{1}{2[\chi(\Delta)]^{\frac{1}{2}}}(A_-\sigma_+ + A_+\sigma_-), \\ \Sigma_2 &= \frac{i}{2[\chi(\Delta)]^{\frac{1}{2}}}(A_+\sigma_- - A_-\sigma_+), \\ \Sigma_3 &= \frac{1}{2}\sigma_z,\end{aligned}\quad (43)$$

where $\chi = \langle n | A_+ A_- | n \rangle$, $|n\rangle$ denotes the eigenstates of A_0 . It is easy to see all these operators commute with Δ and the quasialgebra $\{H, \Sigma_1, \Sigma_2, \Sigma_3\}$ is thus found. This type of *time-dependent* models is therefore proved solvable by working in a sub-Hilbert-space corresponding to the eigenstates of the *time-independent* invariants.

As an illustrative example, we consider the supersymmetric multiphoton Jaynes-Cummings model by means of the invariant theories in the sub-Hilbert-space. In accor-

dance with the Lewis-Riesenfeld invariant theory, the invariant $I(t)$ is often of the form

$$I(t) = c(t)Q^\dagger + c^*(t)Q + b(t)\sigma_z \quad (44)$$

where $c^*(t)$ is the complex conjugation of $c(t)$, and $b(t)$ is real. Substitution of the expressions (44, 18) for $I(t)$ and $H(t)$ into equation (26) leads to the following set of auxiliary equations

$$\begin{aligned}\dot{c} - \frac{1}{i}[c\delta + 2bg] &= 0, & \dot{c}^* + \frac{1}{i}[c^*\delta + 2bg^*] &= 0, \\ \dot{b} + \frac{1}{i}\lambda_m(c^*g - cg^*) &= 0,\end{aligned}\quad (45)$$

where dot denotes the time derivative. The three time-dependent parameters c , c^* and b in $I(t)$ are determined by these three auxiliary equations.

This time-dependent model can be exactly solved by using the invariant-related unitary transformation formulation where the unitary transformation operator is of the form

$$V(t) = \exp[\beta(t)Q - \beta^*(t)Q^\dagger]. \quad (46)$$

with $\beta^*(t)$ being the complex conjugation of $\beta(t)$. With the help of the commutation relations (15), it can be found that, by the complicated and lengthy computations, if $\beta(t)$ and $\beta^*(t)$ satisfy the following equations

$$\begin{aligned}\sin(4\beta\beta^*\lambda_m)^{\frac{1}{2}} &= \frac{\lambda_m(c\beta^* + c^*\beta)}{(4\beta\beta^*\lambda_m)^{\frac{1}{2}}}, \\ \cos(4\beta\beta^*\lambda_m)^{\frac{1}{2}} &= b,\end{aligned}\quad (47)$$

then a time-independent invariant can be obtained as follows

$$I_V \equiv V^\dagger(t)I(t)V(t) = \sigma_z. \quad (48)$$

For convenience, we substitute the time-dependent parameters θ and ϕ for c, c^* and b in $I(t)$ with the help of equation (47), and the results are

$$\begin{aligned}\beta &= -\frac{\theta \exp(-i\phi)}{\lambda_m^{\frac{1}{2}}}, & \beta^* &= -\frac{\theta \exp(i\phi)}{\lambda_m^{\frac{1}{2}}}, \\ c &= -\frac{\sin\theta \exp(-i\phi)}{\lambda_m^{\frac{1}{2}}}, & c^* &= -\frac{\sin\theta \exp(i\phi)}{\lambda_m^{\frac{1}{2}}}.\end{aligned}\quad (49)$$

Thus, the invariant $I(t)$ in (44) can be rewritten

$$I(t) = -\frac{\sin\theta}{\lambda_m^{\frac{1}{2}}}[\exp(-i\phi)Q + \exp(i\phi)Q^\dagger] + \cos\theta\sigma_z. \quad (50)$$

In the meanwhile, under the unitary transformation (46), the Hamiltonian (18) can be transformed into

$$\begin{aligned}H_V(t) &\equiv V^\dagger(t)H(t)V(t) - V^\dagger(t)i\frac{\partial}{\partial t}V(t) \\ &= \omega N - \frac{\omega}{2} + \left\{\frac{\omega}{2}(1 - \cos\theta) - \frac{1}{2}\lambda_m^{\frac{1}{2}}[g \exp(i\phi) \right. \\ &\quad \left. + g^* \exp(-i\phi)] \sin\theta + \frac{\omega - \delta}{2} \cos\theta \right. \\ &\quad \left. - \frac{\dot{\phi}}{2}(1 - \cos\theta)\right\}\sigma_z.\end{aligned}\quad (51)$$

The eigenstates of σ_z corresponding to the eigenvalue $\sigma = +1$ and $\sigma = -1$ are $\binom{1}{0}$ and $\binom{0}{1}$, and the eigenstate of N' is $\binom{|m\rangle}{|m+k\rangle}$ in terms of (17). It follows from equations (35, 37, 38) that two particular solutions of the time-dependent Schrödinger equation of the time-dependent TLMJC model are of the form

$$|\Psi_{m,\sigma=+1}(t)\rangle = \exp\left\{\frac{1}{i}\int_0^t [\dot{\varphi}_{d,\sigma=+1}(t') + \dot{\varphi}_{g,\sigma=+1}(t')]dt'\right\} \times V(t) \binom{|m\rangle}{0} \quad (52)$$

with

$$\begin{aligned} \dot{\varphi}_{d,\sigma=+1}(t') &= \left(m + \frac{k}{2}\right) \omega(t') - \frac{1}{2} \lambda_m^{\frac{1}{2}} \{g(t') \exp[i\phi(t')] \\ &+ g^*(t') \exp[-i\phi(t')]\} \sin \theta(t') \\ &- \frac{\delta(t')}{2} \cos \theta(t') \end{aligned} \quad (53)$$

and

$$\dot{\varphi}_{g,\sigma=+1}(t') = -\frac{\dot{\phi}(t')}{2} [1 - \cos \theta(t')]; \quad (54)$$

and

$$|\Psi_{m,\sigma=-1}(t)\rangle = \exp\left\{\frac{1}{i}\int_0^t [\dot{\varphi}_{d,\sigma=-1}(t') + \dot{\varphi}_{g,\sigma=-1}(t')]dt'\right\} \times V(t) \binom{0}{|m+k\rangle} \quad (55)$$

with

$$\begin{aligned} \dot{\varphi}_{d,\sigma=-1}(t') &= \left(m + \frac{k}{2}\right) \omega(t') + \frac{1}{2} \lambda_m^{\frac{1}{2}} \{g(t') \exp[i\phi(t')] \\ &+ g^*(t') \exp[-i\phi(t')]\} \sin \theta(t') + \frac{\delta(t')}{2} \cos \theta(t') \end{aligned} \quad (56)$$

and

$$\dot{\varphi}_{g,\sigma=-1}(t') = \frac{\dot{\phi}(t')}{2} [1 - \cos \theta(t')]. \quad (57)$$

It should be noted that the above approach to the time-dependent Jaynes-Cummings model is also appropriate for treating the periodic decay and revival of some multiphoton transitions models, which has been investigated by Sukumar and Buck [29].

It is readily verified that the Hamiltonians of the original two-level Jaynes-Cummings model (mono-photon case) [27] and the three-level two-mode mono-photon model possess the $SU(2)$ and $SU(3)$ Lie-algebraic structure, respectively, in the sub-Hilbert-space corresponding to a particular eigenvalue of the conserved generator (the *time-independent* invariant). It follows from equations (52–57) that the solution of the time-dependent case of $SU(2)$ Jaynes-Cummings model is easily obtained by taking $k = 1$, where k denotes the number of photons mediating in the process of atomic transitions. Since

Shumovsky *et al.* have considered the three-level two-mode multiphoton Jaynes-Cummings model [41] whose Hamiltonian is time-independent, we think it is also of interest to exactly solve the time-dependent supersymmetric three-level two-mode multiphoton Jaynes-Cummings model by means of these invariant theories.

5 Concluding remarks

(1) On the basis of the fact that all simple three-generator algebras are either isomorphic to the algebra $sl(2, C)$ or to one of its real forms, exact solutions of the time-dependent Schrödinger equation of all three-generator systems and models in quantum optics, nuclear physics, solid state physics, molecular and atomic physics as well as laser physics are provided by making use of both the Lewis-Riesenfeld invariant theory and the invariant-related unitary transformation formulation. We uses the unitary transformation and obtain the explicit expression for the time-evolution operator, instead of the formal solution that is related to the chronological product.

(2) Since it appears only in systems with time-dependent Hamiltonian, the geometric phase factor would be easily studied if the exact solutions of time-dependent systems had been obtained. In the adiabatic limit, *i.e.*, if the parameter a is taken to be time-independent, then the geometric phase in a cycle associated with $b(t)$ can be rewritten as $\varphi_g(T) = (\lambda/m)[2\pi(1 - \cos a)]$, where $2\pi(1 - \cos a)$ is an expression for the solid angle over the parameter space of the invariant. It is well known that this phase is just the Berry's adiabatic geometric phase (*i.e.*, Berry's non-integral phase), which is found by Berry in the quantum adiabatic process in 1984 [1]. In this paper, however, we obtain the non-adiabatic non-cyclic geometric phase in time-dependent quantum systems; Berry's phase is therefore only the particular case of ours presented in this paper. In view of above discussions, the invariant-related unitary transformation formulation is a useful tool for treating the geometric phase factor and the time-dependent Schrödinger equation. This formulation replaces the eigenstates of the *time-dependent* invariants with those of the *time-independent* invariants through the unitary transformation and thus obtain the explicit solutions, rather than the formal solutions associated with the chronological product, of time-dependent quantum systems.

(3) It is known that the time-dependent Schrödinger equation can be solved if its Hamiltonian is constructed in terms of the generators of a certain Lie algebra. For some quantum systems whose time-dependent Hamiltonians possess no quasialegbraic structures, we show that the exact solutions can also be obtained by working in a sub-Hilbert-space corresponding to a particular eigenvalue of the conserved generator (*i.e.*, the *time-independent* invariant that commutes with the *time-dependent* Hamiltonian). In Section 4, we obtain the complete set of exact solutions of the *time-dependent* supersymmetric multiphoton Jaynes-Cummings model in the sub-Hilbert-space corresponding to the *time-independent invariant* N' .

Apparently, the method presented in this paper is also applicable to the algebraic structure whose number of generators is more than three. Additionally, it should be pointed out that the time-dependent Schrödinger equation is often considered in literature, whereas less attention is paid to the time-dependent Klein-Gordon equation. Since it can govern the time evolution of some scalar fields, we think that it gets less attentions than it deserves. Work in this direction is under consideration and will be published elsewhere.

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Note added in proof

The expression for geometric phase is analogous to the gravitomagnetic vector potential [38] of a hypothetical gravitomagnetic monopole. Thus, a scalar particle moving in a gravitomagnetic field of gravitomagnetic monopole may give rise to an Aharonov-Carni phase that is in exact analogy with the Berry's phase, which a spinning particle will also acquire due to the time-dependent spin-rotation coupling in a non-inertial frame of reference. This geometric phase (so-called spin-rotation geometric phase) itself appears to possess much physical significance and deserves investigations in more detail.

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